Lecture 7
Carrier-based Differential Positioning. Ambiguity Resolution Techniques

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Linear model for Differential Positioning

**Single difference**

\[ (\bullet)_r^j \equiv \Delta(\bullet)_r^j = (\bullet)_u^j - (\bullet)_r^j \]

\[ P_{ru}^j = \rho_{ru}^j + c \delta t_{ru} + T_{ru}^j + I_{ru}^j + K_{ru} + \nu_{p,ru}^j \]

\[ L_{ru}^j = \rho_{ru}^j + c \delta t_{ru} + T_{ru}^j - I_{ru}^j + \lambda \omega_{ru}^j + \lambda N_{ru}^j + b_{ru} + \nu_{l,ru}^j \]

**Double difference**

\[ (\bullet)_{ru}^{jk} \equiv \nabla \Delta(\bullet)_{ru}^{jk} = (\bullet)_u^k - (\bullet)_r^k - (\bullet)_u^j - (\bullet)_r^j \]

\[ P_{ru}^k = \rho_{ru}^k + c \delta t_{ru} + T_{ru}^k + I_{ru}^k + K_{ru} + \nu_{p,ru}^k \]

\[ P_{ru}^j = \rho_{ru}^j + c \delta t_{ru} + T_{ru}^j + I_{ru}^j + K_{ru} + \nu_{p,ru}^j \]

\[ P_{ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{jk} + I_{ru}^{jk} + \nu_{p,ru}^{jk} \]

The same for carrier:

\[ L_{ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{jk} - I_{ru}^{jk} + \lambda \omega_{ru}^{jk} + \lambda N_{ru}^{jk} + \nu_{l,ru}^{jk} \]

Now are cancelled:
- Receiver clock
- Receiver code instrumental delays
- Receiver carrier instrumental delays

\[ \rightarrow \text{Carrier ambiguities } N\text{ are integer} \]
Single-Difference of measurements (corrected by geometric range!!)

\[ \Delta(L_1 - \rho) \equiv L_{1ru}^{sat} - \rho_{ru}^{sat} \]

\[ \Delta(P_1 - \rho) \equiv P_{ru}^{sat} - \rho_{ru}^{sat} \]

Dif. Wind-up: Very small

Dif. Receiver clock: Main variations Common for all satellites

Dif. Tropo. and Iono.: Small variations

Dif. Instrumental delays and carrier ambiguities:

\[ \Delta(L_1 - \rho) = c \delta t_{ru} + T_{ru}^j - I_{ru}^j + \lambda \omega_{ru}^j + \lambda N_{ru}^j + b_{ru} + \nu_{Lru}^j \]

\[ \Delta(P_1 - \rho) = c \delta t_{ru} + T_{ru}^j + I_{ru}^j + K_{ru} + \nu_{p_{ru}}^j \]

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Double-Difference of measurements (corrected by geometric range!!)

Dif. wind-up: negligible

\[ \nabla \Delta L_1 - \nabla \Delta \rho \equiv L_{ru}^{sat} - \rho_{ru}^{sat} \]

\[ \nabla \Delta P_1 - \nabla \Delta \rho \equiv P_{ru}^{sat} - \rho_{ru}^{sat} \]

Carrier ambiguities: constant

Dif. Tropo. and Iono. :
Small variations

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Linear model for Differential Positioning

**Single difference** \((\odot)^j_{ru} \equiv \Delta(\odot)^j_{ru} = (\odot)^j_u - (\odot)^j_r\)

\[ P^j_{ru} = \rho^j_{ru} + c \delta t_{ru} + T^j_{ru} + I^j_{ru} + K_{ru} + \nu^j_{\rho_{ru}} \]

\[ L^j_{ru} = \rho^j_{ru} + c \delta t_{ru} + T^j_{ru} - I^j_{ru} + \lambda \omega^j_{ru} + \lambda N^j_{ru} + b_{ru} + \nu^j_{L_{ru}} \]

where:
\[ \rho^j_{ru} = \rho^j_{0ru} - \hat{\rho}^j_{0u} \cdot \Delta r_{ru} - \hat{\rho}^j_{0ru} \cdot \varepsilon_{\text{site}} + \hat{\rho}^j_{0ru} \cdot \varepsilon_{\text{eph}} \]

**Double difference**

\[(\odot)^j_{rk} \equiv \nabla \Delta(\odot)^j_{ru} = (\odot)^k_{ru} - (\odot)^j_{ru} = (\odot)^k_u - (\odot)^k_r - [(\odot)^j_u - (\odot)^j_r]\]

\[ P^j_{ru} = \rho^j_{ru} + T^j_{ru} + I^j_{ru} + \nu^j_{\rho_{ru}} \]

\[ L^j_{ru} = \rho^j_{ru} + T^j_{ru} - I^j_{ru} + \lambda \omega^j_{ru} + \lambda N^j_{ru} + \nu^j_{L_{ru}} \]

where:
\[ \rho^j_{ru} = \rho^j_{0ru} - \hat{\rho}^j_{0u} \cdot \Delta r_{ru} - \hat{\rho}^j_{0u} \cdot \varepsilon_{\text{eph}} + \hat{\rho}^k_{0u} \cdot \varepsilon^k_{eph} - \hat{\rho}^j_{0u} \cdot \varepsilon^j_{eph} \]

being:
\[ \rho^j_{0ru} \equiv \rho^j_{0u} - \rho^j_{0r} ; \quad \Delta r_{ru} \equiv \Delta r_u - \Delta r_r \]
**Exercise:**

Consider the Single Differences of geometric range:

\[
\rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \epsilon_{site} + \hat{\rho}_{0ru}^j \cdot \epsilon_{eph}
\]

where \( \rho_{ru}^j = \rho_u^j - \rho_r^j \)

Show that the Double Differences are given by:

\[
\rho_{ru}^{jk} = \rho_{0ru}^{jk} - \hat{\rho}_{0u}^{jk} \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^{jk} \cdot \epsilon_{site} + \hat{\rho}_{0ru}^{jk} \cdot \epsilon_{eph} - \hat{\rho}_{0ru}^j \cdot \epsilon_{eph} - \hat{\rho}_{0ru}^j \cdot \epsilon_{eph}
\]

being:

\[
\rho_{0ru}^{jk} \equiv \rho_{0u}^{jk} - \rho_{0r}^{jk} ; \quad \Delta r_{ru} \equiv \Delta r_u - \Delta r_r
\]
Linear model for Differential Positioning

\[
(\bullet)^{jk}_{ru} \equiv \nabla \Delta (\bullet)^{jk}_{ru} = (\bullet)^{k}_{ru} - (\bullet)^{j}_{ru} = (\bullet)^{k}_{u} - (\bullet)^{k}_{r} - \left[ (\bullet)^{j}_{u} - (\bullet)^{j}_{r} \right]
\]

\[
P_{jk}^{ru} = \rho_{jk}^{ru} + T_{jk}^{ru} + I_{jk}^{ru} + \nu_{jk}^{ru}
\]

\[
L_{jk}^{ru} = \rho_{jk}^{ru} + T_{jk}^{j} - I_{jk}^{j} + \lambda \sigma_{jk}^{ru} + \lambda N_{jk}^{ru} + \nu_{L}^{jk}
\]

where:

\[\rho_{jk}^{ru} = \rho_{0ru}^{jk} - \hat{\rho}_{0u}^{jk} \cdot \Delta r_{ru} - \hat{\rho}_{0u}^{jk} \cdot \epsilon_{site} + \hat{\rho}_{0u}^{k} \cdot \epsilon_{eph} - \hat{\rho}_{0u}^{j} \cdot \epsilon_{eph} \]

For short baselines (e.g. up to 10 km) and if the reference station coordinates are accurately known, we can assume:

\[T_{ru}^{jk} \equiv 0 ; I_{ru}^{jk} \equiv 0 ; \sigma_{ru}^{jk} \equiv 0\]

\[\hat{\rho}_{0ru}^{j} \cdot \epsilon_{eph} \equiv 0\]

\[\epsilon_{site} \equiv 0 \Rightarrow \Delta r_{ru} \equiv \Delta r_{u}\]

Note for baselines up to 10 km
the range error of broadcast orbits is less than 1 cm
(assuming \(\epsilon_{eph} \equiv 10\) m).

With these simplifications, we have:

\[
P_{jk}^{ru} - \rho_{0ru}^{jk} = -\hat{\rho}_{0u}^{jk} \cdot \Delta r_{u} + \nu_{P}^{jk}
\]

\[
L_{jk}^{ru} - \rho_{0ru}^{jk} = -\hat{\rho}_{0u}^{jk} \cdot \Delta r_{u} + \lambda N_{ru}^{jk} + \nu_{L}^{jk}
\]

Remark: \[P_{jk}^{ru} - \rho_{0ru}^{jk} = P_{jk}^{ru} - \rho_{0u}^{jk} - \left( P_{r}^{jk} - \rho_{0r}^{jk} \right)\]
As with the SD, the left hand side of previous equations can be spitted in two terms: one associated to the reference station and the other to the user. Then, the differential corrections can be computed for code and carrier as:

\[
P_{\text{RC}}^{jk} = \rho_0^{jk} - P^j_r; \quad P_{\text{RC}}^{Lk} = \rho_0^{jk} - L^j_r
\]

- The user applies this differential correction to remove/mitigate common errors:

\[
\begin{align*}
P^j_u - \rho_0^{jk} + P_{\text{RC}}^{jk} &= -\hat{\rho}^{jk}_0 \cdot \Delta r^j_u + \nu^{jk}_{p ru} \\
L^j_u - \rho_0^{jk} + P_{\text{RC}}^{Lk} &= -\hat{\rho}^{jk}_0 \cdot \Delta r^j_u + \lambda N^j_{ru} + \nu^{jk}_{L ru}
\end{align*}
\]

Where the carrier ambiguities \(N\) are integer numbers and must be estimated together with the user solution.

For larger distances, the atmospheric propagation effects (troposphere, ionosphere) must be removed with accurate modelling. Wide area users will require also orbit corrections.

\[
\text{Remark: } P^{jk}_{ru} - \rho_0^{jk} = P^j_u - \rho_0^{jk} - \left( P^j_r - \rho_0^{jk} \right)
\]
Differential code and carrier positioning

The user applies this differential correction to remove/mitigate common errors

\[
P_{u}^{jk} - \rho_{0u}^{jk} + \text{PRC}_{P}^{jk} = -\hat{\rho}_{0u}^{jk} \cdot \Delta \mathbf{r}_{u} + \nu_{p}^{jk}
\]

\[
L_{u}^{jk} - \rho_{0u}^{jk} + \text{PRC}_{L}^{jk} = -\hat{\rho}_{0u}^{jk} \cdot \Delta \mathbf{r}_{u} + \lambda \mathbf{N}_{ru}^{jk} + \nu_{L}^{jk}
\]

where

\[
\hat{\rho}_{0u}^{jk} \equiv \hat{\rho}_{0u}^{k} - \hat{\rho}_{0u}^{j}
\]

The previous system for navigation equations is written in matrix notation as:

\[
\begin{bmatrix}
\text{Pref}_{P_{R,1}} \\
\text{Pref}_{L_{R,1}} \\
\vdots \\
\text{Pref}_{P_{R,n-1}} \\
\text{Pref}_{L_{R,n-1}}
\end{bmatrix}
= 

\begin{bmatrix}
-\left(\hat{\rho}_{0u}^{1} - \hat{\rho}_{0u}^{R}\right)^T & 0 & \cdots & 0 \\
-\left(\hat{\rho}_{0u}^{1} - \hat{\rho}_{0u}^{R}\right)^T & 1 & \cdots & 0 \\
-\left(\hat{\rho}_{0u}^{n-1} - \hat{\rho}_{0u}^{R}\right)^T & 0 & \cdots & 0 \\
-\left(\hat{\rho}_{0u}^{n-1} - \hat{\rho}_{0u}^{R}\right)^T & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\Delta \mathbf{r}_{u} \\
\lambda \mathbf{N}_{R,1}^{R} \\
\vdots \\
\lambda \mathbf{N}_{R,n-1}^{R} \\
\end{bmatrix}
\]

where

\[
\text{Pref}_{P_{R,k}}^{R} \equiv P_{u}^{R,k} - \rho_{0u}^{R,k} + \text{PRC}_{P}^{R,k}
\]

\[
\text{Pref}_{L_{R,k}}^{R} \equiv L_{u}^{R,k} - \rho_{0u}^{R,k} + \text{PRC}_{L}^{R,k}
\]

Carrier ambiguities
- The **reference station** with known coordinates, computes differential corrections:

\[
P_{jk}^{PRC} = \rho_{0r}^{jk} - P_r^{jk} ; \quad P_{jk}^{PRC} = \rho_{0r}^{jk} - L_r^{jk}
\]

- The **user** receiver applies these corrections to its own measurements to remove SIS errors and improve the positioning accuracy.
Correlations among the DD Measurements

We assume uncorrelated measurements (both code and carrier). Then, the covariance matrix is diagonal:

$$\mathbf{P}_P = \sigma_P^2 \mathbf{I} \quad \mathbf{P}_L = \sigma_L^2 \mathbf{I} \quad \text{where, we can assume:} \quad \sigma_P \approx 50cm \quad ; \quad \sigma_L \approx 5mm$$

Let $X$ represent the code $P$ or the Carrier $L$ measurement.

- The **single difference (SD)** and its covariance matrix can be computed as:

$$\begin{bmatrix} X^k_u \\ X^j \\ X^k_r \\ X^j_r \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X^k_u \\ X^k_r \\ X^j_u \\ X^j_r \end{bmatrix} \quad ; \quad \mathbf{P}_{X_{SD}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 & 0 & 0 \\ 0 & \sigma_X^2 & 0 & 0 \\ 0 & 0 & \sigma_X^2 & 0 \\ 0 & 0 & 0 & \sigma_X^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = 2\sigma_X^2 \mathbf{I}$$

Thence, if the measurements are uncorrelated, so are they in single differences, but the noise is twice!
Correlations among the DD Measurements

• Now, the double difference (DD) and its covariance matrix can be computed as:

\[
\begin{bmatrix}
X_{ru}^{jk} \\
X_{ru}^{jl}
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
X_{ru}^k \\
X_{ru}^j \\
X_{ru}^l
\end{bmatrix};
\]

\[
P_{X_{DD}} =
\begin{bmatrix}
1 & -1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
2\sigma_X^2 & 0 & 0 \\
0 & 2\sigma_X^2 & 0 \\
0 & 0 & 2\sigma_X^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1 & -1 \\
0 & 1
\end{bmatrix} = 2\sigma_X^2
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

Thence, even if the original measurements are uncorrelated, the double differences are correlated.

Note: The removal of the relative “User”—“Reference-station” common bias (e.g. relative receiver clock) in DD is done at the expense of one observation and the introduction of a correlation between measurements.
Single and double differences of receivers/satellites

\[ \Delta \equiv \Delta - \Delta = \nabla - \nabla \]

Receiver errors affecting both satellites are removed (e.g. Receiver clock)

\[ \Delta \equiv \Delta_{rov} - \Delta_{ref} \]

SIS errors affecting both receivers are removed (e.g. Satellite clocks,...)

Receiver errors common for all satellites do not affect positioning (as they are assimilated in the receiver clock estimate). Thence:

- Only residual errors in single differences between sat. affect absolute posit.
- Only residual errors in double differences between sat. and receivers affect relative positioning.

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Single and double differences of receivers/satellites

\[ \nabla \Delta \equiv \nabla - \Delta R \]

Receiver errors affecting both satellites are removed (e.g. Receiver clock)

\[ \Delta \equiv \Delta_{rov} - \Delta_{ref} \]

SIS errors affecting both receivers are removed (e.g. Satellite clocks,...)

When comparing SD and DD one might suggest that in the DD formulation there is even further error reduction, positively influencing the results in positioning. This is however not true, since in the SD case the mean value of unmodelled effects is absorbed by the receiver clock. If the DD correlations are taken into account, the positioning results in both cases are the same. However the DD formulation has the advantage that it allows the direct estimation of the ambiguities.

\[ \Delta \nabla \equiv \Delta \Delta_{rov} - \Delta_{ref} \]
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Relative Positioning

The following relationship can be obtained from the figure, where we assume that the baseline is shorter than the distance to the satellite by orders of magnitude:

\[ \rho_{ru}^j = \rho_u^j - \rho_r^j = -\hat{\rho}_u^j \cdot r_{ru} \]

Applying the same scheme to a second satellite “k”

\[ \rho_{ru}^k = \rho_u^k - \rho_r^k = -\hat{\rho}_u^k \cdot r_{ru} \]

Thence, the double differences of ranges are:

\[ \rho_{ru}^{jk} = \rho_{ru}^k - \rho_{ru}^j = -\left(\hat{\rho}_u^k - \hat{\rho}_u^j\right) \cdot r_{ru} = -\hat{\rho}_u^{jk} \cdot r_{ru} \]
Relative Positioning

Thence, the double differences of ranges are:

\[ \hat{\rho}_{jk}^l = \hat{\rho}_{k}^l - \hat{\rho}_{j}^l = -(\hat{\rho}_{u}^k - \hat{\rho}_{u}^j) \cdot r_{ru} = -\hat{\rho}_{u}^j \cdot r_{ru} \]

As commented before, for short baselines (e.g. less than 10km), we can assume that ephemeris and propagation errors cancel, thence:

\[ P_{ru}^j = \rho_{ru}^j + T_{ru}^j + I_{ru}^j + \nu_{Pru}^j \]
\[ L_{ru}^j = \rho_{ru}^j + T_{ru}^j - I_{ru}^j + \lambda \omega_{ru}^j + \lambda N_{ru}^j + \nu_{Lru}^j \]

Note that these equations allows a direct estimation of the baseline, without needing an accurate knowledge of the reference station coordinates.
Relative Positioning

The previous system for navigation equations is written in matrix notation as:

\[
\begin{pmatrix}
P_{\text{r},u}^{R,1} \\ L_{\text{r},u}^{R,1} \\ \vdots \\ P_{\text{r},u}^{R,n-1} \\ L_{\text{r},u}^{R,n-1}
\end{pmatrix}
= \begin{pmatrix}
-\left(\hat{\rho}_u^{1} - \hat{\rho}_u^{R}\right)^T & 0 & \cdots & 0 \\
-\left(\hat{\rho}_u^{n-1} - \hat{\rho}_u^{R}\right)^T & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\left(\hat{\rho}_u^{n-1} - \hat{\rho}_u^{R}\right)^T & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\lambda N_{\text{r},u}^{R,1} \\
\lambda N_{\text{r},u}^{R,2} \\
\vdots \\
\lambda N_{\text{r},u}^{R,n-1}
\end{pmatrix}
\]

where

\[
\hat{\rho}_u^{jk} = \hat{\rho}_u^k - \hat{\rho}_u^j
\]

DD Code and Carrier measurements

Carrier ambiguities

Baseline vector

Satellite-j

Reference Station

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Double-Difference of measurements

\[ \nabla \Delta L_1 \equiv L_{1ru}^{sat} \]

\[ \nabla \Delta P_1 \equiv P_{1ru}^{sat} \]

Variation of the Baseline projection over the unit Line-Of-Sight vector

\[ \rho_{ru} = -\hat{\rho}_u \cdot \mathbf{r}_{ru} \]
Double-Difference of measurements

\[ \nabla \Delta L_1 \equiv L_{1ru}^{sat} \]

\[ \nabla \Delta P_1 \equiv P_{1ru}^{sat} \]
Relative Positioning

In this approach, the reference station broadcast its time-tagged code and carrier measurements, instead of the computed differential corrections.

Thence, the user can form the double differences of its own measurements with those of the reference receiver, satellite by satellite, and estimate its position relative to the reference receiver.

Notice that, the baseline can be estimated without needing an accurate knowledge of reference the station coordinates. Of course, the knowledge of the reference station coordinates would allow the user to compute its absolute coordinates.
Relative Positioning

Time synchronization issues:

There is an important and subtitle difference between the previous approach of relative positioning (which does not need to know the reference station coordinates) and the differential positioning approach based on the knowledge of the reference station coordinates.

• The differential corrections vary slowly, and its useful life can be up to several minutes with S/A=off.

• But, the measurements change much faster. The range rate can be up to 800m/s and, thence, a synchronizer error of 1 millisecond can lead up to more than 1/2 meter of error.

• As commented before, real-time implementation entails also latencies, that can be up to 2 seconds, thence, a extrapolation technique must be applied to the measurements to reduce error due to latency and epoch mismatch (to <1cm if ambiguities are intended to be fixed).
PRC (from GODS) : 2013 02 21

**PRC**

RRC (from GODS) : 2013 02 21

**RRC**

Receiver: JAVAD TRE_G3TH DELTA3.3.12

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\[ dt_{rec} \approx 300 \text{ km/450 s} = 667 \text{ m/s} \]

\[ dL_1 \approx d\rho + dt_{rec} \]

1 ms jump of receiver clock adjust

300Km=1ms

\[ dt_{rec} \approx 660 \text{ m/s} \]

\[ d\rho \text{ up to } \sim 800 \text{ m/s} \]
COMMENTS

Real-Time implementation entails delays in data transmission, which can reach up to 1 or 2 s.

- Differential corrections vary slowly and its useful life is of several minutes (S/A=off)
- But, the measurements change much faster:
  - The range rate $\frac{d\rho}{dt}$ can be up to 800 m/s and, therefore, the range can change by more than half a meter in 1 millisecond. Moreover the receiver clock offset can be up to 1 millisecond (depending on the receiver configuration).
  - Thence, the reference station measurements must be:
    - **Synchronized** to reduce station clock mismatch: station clock can be estimated to within 1 µs $\Rightarrow \varepsilon_{dt_{sta}} < 1$ mm
    - **Extrapolated** to reduce error due to latency: carrier can be extrapolated with error $< 1$ cm.

\[ RRC = \frac{\Delta PRC}{\Delta t} \]

\[ dL_1 \approx d\rho + dt_{rec} \]

\[ dt_{rec} \sim 660 \text{ m/s} \]
\[ d\rho \text{ up to } \sim 800 \text{ m/s} \]
**Exercise:**

Demonstrate the following relationship between the baseline and the differential range [*]:

\[
\rho_{ru}^j = \rho_u^j - \rho_r^j = -\left(\frac{2\rho_u^j + \mathbf{r}_{ru}}{\|\rho_u^j\| + \|\rho_u^j + \mathbf{r}_{ru}\|}\right) \cdot \mathbf{r}_{ru}
\]

*This result is from [RD-7]*

**Comments:**

The previous expression can be written as:

\[
\rho_u^j - \rho_r^j = -\left(\omega^j \hat{\mathbf{e}}^j\right) \cdot \mathbf{r}_{ru} \quad \text{with} \quad \omega^j = \frac{2\rho_u^j + \mathbf{r}_{ru}}{\|\rho_u^j\| + \|\rho_u^j + \mathbf{r}_{ru}\|} \quad \hat{\mathbf{e}}^j = \frac{\rho_u^j + \mathbf{r}_{ru}}{2} \quad \frac{\rho_u^j + \mathbf{r}_{ru}}{2}
\]

- Taking \( \mathbf{r}_{ru} = 0 \) in \( \omega^j \) and \( \hat{\mathbf{e}}^j \) leads to the approximate expression previously found.

- \( \omega^j \) and \( \hat{\mathbf{e}}^j \) depend on the baseline \( \mathbf{r}_{ru} \), which is the vector to estimate. Nevertheless, it is not very sensitive to changes in such baseline and can be computed iteratively, computing the navigation solution starting from \( \mathbf{r}_{ru} = 0 \).

*Comments:*
Relative Positioning

Solution

Consider the following relations:

\[ \mathbf{r}_{ru} = \mathbf{p}_r^j - \mathbf{p}_u^j \]

\[ \hat{\mathbf{e}}^j = \frac{\mathbf{p}_u^j + \mathbf{r}_{ru} / 2}{\| \mathbf{p}_u^j + \mathbf{r}_{ru} / 2 \|} = \frac{\mathbf{p}_r^j + \mathbf{p}_u^j}{2 \mathbf{p}_u^j + \mathbf{r}_{ru}} \]

\[
\left( \| 2 \mathbf{p}_u^j + \mathbf{r}_{ru} \| \hat{\mathbf{e}}^j \right) \cdot \mathbf{r}_{ru} = \| \mathbf{p}_r^j \|^2 - \| \mathbf{p}_u^j \|^2 = \\
= \left( \| \mathbf{p}_r^j \| - \| \mathbf{p}_u^j \| \right) \left( \| \mathbf{p}_r^j \| + \| \mathbf{p}_u^j \| \right) \\
= ( \mathbf{p}_r^j - \mathbf{p}_u^j ) ( \| \mathbf{p}_u^j + \mathbf{r}_{ru} \| + \| \mathbf{p}_u^j \| )
\]

Then:

\[ \mathbf{p}_u^j - \mathbf{p}_r^j = - \left( \omega^j \hat{\mathbf{e}}^j \right) \cdot \mathbf{r}_{ru} \]

with:

\[ \omega^j = \frac{\| 2 \mathbf{p}_u^j + \mathbf{r}_{ru} \|}{\| \mathbf{p}_u^j \| + \| \mathbf{p}_u^j + \mathbf{r}_{ru} \|} \]

\[ \omega^j \hat{\mathbf{e}}^j = \frac{2 \mathbf{p}_u^j + \mathbf{r}_{ru}}{\| \mathbf{p}_u^j \| + \| \mathbf{p}_u^j + \mathbf{r}_{ru} \|} \]
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The Role of Geometric Diversity: **Triple differences**

Let us consider again the problem of relative positioning for short baselines. We have previously found the following equation for DD carrier measurements, assuming short baselines (e.g. < 10km)

\[
L_{ru}^{jk} = -\left( \hat{\rho}_u^k - \hat{\rho}_u^j \right) \cdot \mathbf{r}_{ru} + \lambda N_{ru}^{jk} + \nu_{Lru}^{jk}
\]

As the ambiguities are constant along continuous carrier phase arcs, an option could be to take differences on time. Thence, if the user and reference receiver are stationary we can write the “**triple differences**” as:

\[
\delta L_{ru}^{jk} = -\left( \delta \hat{\rho}_u^k - \delta \hat{\rho}_u^j \right) \cdot \mathbf{r}_{ru} + \delta \nu_{Lru}^{jk} \quad \Rightarrow
\]

\[
\begin{bmatrix}
\delta L_{ru}^{12} \\
\delta L_{ru}^{13} \\
\vdots \\
\delta L_{ru}^{1K}
\end{bmatrix} = \begin{bmatrix}
-\left( \delta \hat{\rho}_u^2 - \delta \hat{\rho}_u^1 \right)^T \\
-\left( \delta \hat{\rho}_u^3 - \delta \hat{\rho}_u^1 \right)^T \\
\vdots \\
-\left( \delta \hat{\rho}_u^K - \delta \hat{\rho}_u^1 \right)^T
\end{bmatrix} \mathbf{r}_{ru} + \tilde{\nu}
\]

where:

\[
\delta L_{ru}^{jk} \equiv L_{ru}^{jk}(t_2) - L_{ru}^{jk}(t_1)
\]

For simplicity, we assign \((j=1)\) to the reference satellite
The Role of Geometric Diversity: **Triple differences**

\[
\delta L_{ru}^{jk} = -\left( \delta \hat{\rho}_u^k - \delta \hat{\rho}_u^j \right) \cdot \mathbf{r}_{ru} + \delta \nu_{L_{ru}}^{jk} \quad \Rightarrow \\
\begin{bmatrix}
\delta L_{ru}^{12} \\
\delta L_{ru}^{13} \\
\vdots \\
\delta L_{ru}^{1K}
\end{bmatrix} = \\
\begin{bmatrix}
-(\delta \hat{\rho}_u^2 - \delta \hat{\rho}_u^1)^T \\
-(\delta \hat{\rho}_u^3 - \delta \hat{\rho}_u^1)^T \\
\vdots \\
-(\delta \hat{\rho}_u^K - \delta \hat{\rho}_u^1)^T
\end{bmatrix} \mathbf{r}_{ru} + \tilde{\nu}
\]

Now, we have a “clean” equations system involving only the baseline vector to estimate. But the geometry is very weak (the associated DOP will be large number) and the position estimates will be in general worse than those from double differences.
Estimation of position and change in position: the role of Geometric Diversity

Let us now consider a simple model for the estimation of the relative position vector from SD carrier measurements, assuming short baselines (e.g. <10km):

\[ L^j_{ru} = \rho^j_{ru} + c \delta t_{ru} + \lambda N^j_{ru} + b_{ru} + \nu^j_{\lambda ru} \]

\[ \rho^j_{ru} = \rho^j_u - \rho^j_r = -\hat{\rho}^j_u \cdot r_{ru} \]

\[ L^j_{ru} = -\hat{\rho}^j_u \cdot r_{ru} + d_{ru} + \lambda N^j_{ru} + \nu^j_{\lambda ru} \]

where \[ d_{ru} \equiv c \delta t_{ru} + b_{ru} \]

\[
\begin{bmatrix}
L^1_{ru} \\
L^2_{ru} \\
\vdots \\
L^K_{ru}
\end{bmatrix} = \begin{bmatrix}
(-\hat{\rho}^1_u)^T & 1 \\
(-\hat{\rho}^2_u)^T & 1 \\
(-\hat{\rho}^K_u)^T & 1
\end{bmatrix} \begin{bmatrix}
r_{ru} \\
d_{ru}
\end{bmatrix} + \begin{bmatrix}
\lambda N^1_{ru} \\
\lambda N^2_{ru} \\
\vdots \\
\lambda N^K_{ru}
\end{bmatrix} + \nu
\]

This is from [RD-3]
Estimation of position and change in position: the role of Geometric Diversity

Previous system can be arranged as:

\[
\begin{bmatrix}
L_{ru}^1 \\
L_{ru}^2 \\
\vdots \\
L_{ru}^K
\end{bmatrix} =
\begin{bmatrix}
(-\hat{p}^1_u)^T & 1 \\
(-\hat{p}^2_u)^T & 1 \\
\vdots \\
(-\hat{p}^K_u)^T & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{r}_{ru} \\
\mathbf{d}_{ru}
\end{bmatrix} +
\begin{bmatrix}
\lambda N_{ru}^1 \\
\lambda N_{ru}^2 \\
\vdots \\
\lambda N_{ru}^K
\end{bmatrix} + \mathbf{v}
\]

Considering now differences between two epochs \(t_0\) and \(t_1\), and assuming no cycle-slips:

\[
\mathbf{L}_{ru}(t_1) - \mathbf{L}_{ru}(t_0) = \mathbf{G}(t_1)
\begin{bmatrix}
\delta \mathbf{r}_{ru}(t_1) \\
\delta \mathbf{d}_{ru}(t_1)
\end{bmatrix} + \mathbf{G}(t_1) - \mathbf{G}(t_0)
\begin{bmatrix}
\mathbf{r}_{ru}(t_0) \\
\mathbf{d}_{ru}(t_0)
\end{bmatrix} + \mathbf{v}
\]

Estimation of changes in baseline vector and clock bias is tied to the geometry matrix at time \(t_1\). This can be well determined.

Estimation of absolute value of baseline vector is tied to the change in geometry matrix at time \(t_1\). This would be poor determined if such change is not significant.

\(d_{ru}(t_0)\) cannot be estimated at all!
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   - Least-Squares Ambiguity Search Technique.
   - LAMBDA Method.
Ambiguity resolution Techniques

As a driven problem to study the ambiguity fixing, we will consider the problem of differential positioning in DD for short baselines (e.g. < 10 km). In general we will consider that we have Code and Carrier measurements in different frequencies \((q=1,2\ldots)\), i.e. \(P_1, P_2, L_1, L_2\ldots\).

\[
\begin{align*}
P^{jk}_{q,ru} &= \rho^{jk}_{ru} + T^{jk}_{ru} + I^{jk}_{q,ru} + v^{jk}_{P,q,ru} ; \quad q = 1,2\ldots \\
L^{jk}_{q,ru} &= \rho^{jk}_{ru} + T^{j}_{ru} - I^{jk}_{q,ru} + \lambda_q \omega^{jk}_{ru} + \lambda N^{jk}_{q,ru} + v^{jk}_{L,q,ru}
\end{align*}
\]

To simplify notation, when different frequencies are considered, we will remove the subscript “\(ru\)”.

\[
\begin{align*}
\rho^{jk}_{ru} + T^{j}_{ru} - I^{jk}_{q,ru} + \lambda \omega^{jk}_{ru} + \lambda N^{jk}_{q,ru} + v^{jk}_{L,q,ru}
\end{align*}
\]

As a driven problem to study the ambiguity fixing, we will consider the problem of differential positioning in DD for short baselines (e.g. < 10 km). In general we will consider that we have Code and Carrier measurements in different frequencies \((q=1,2\ldots)\), i.e. \(P_1, P_2, L_1, L_2\ldots\).

\[
\begin{align*}
P^{jk}_{q,ru} &= \rho^{jk}_{ru} + v^{jk}_{P,q,ru} ; \quad q = 1,2\ldots \\
L^{jk}_{q,ru} &= \rho^{jk}_{ru} + \lambda_q N^{jk}_{q,ru} + v^{jk}_{L,q,ru}
\end{align*}
\]

Take the highest elevation as the reference satellite to minimize measurement error.

We assume the following measurement errors:

\[
\begin{align*}
\sigma_{P_q} &\approx 0.5\text{ m} & \sigma_{\rho^{jk}_{P,q}} &\approx 1\text{ m} \\
\sigma_{L_q} &\approx 0.5\text{ cm} & \sigma_{\rho^{jk}_{L,q}} &\approx 1\text{ cm}
\end{align*}
\]

As commented before, the ambiguity terms are integer numbers, and we can take benefit of this property to fix such ambiguities applying integer ambiguity resolution techniques.
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Resolving ambiguities one at a Time

A simple trial would be (for instance using L1 and P1):

\[ P_{1j}^{jk} = \rho_{jk}^{P1} + \nu_{P1}^{jk} \]

\[ L_{1j}^{jk} = \rho_{jk}^{L1} + \lambda_{1} N_{1j}^{jk} + \nu_{L1}^{jk} \]

\[ \rightarrow L_{1j}^{jk} - P_{1j}^{jk} = \lambda_{1} N_{1j}^{jk} + \nu_{P1}^{jk} \rightarrow \]

\[ \hat{N}_{1j}^{jk} = \left[ \frac{L_{1j}^{jk} - P_{1j}^{jk}}{\lambda_{1}} \right] \text{roundoff} \]

\[ \lambda_{1} \approx 20 \text{ cm} \]

\[ \sigma_{P_{1j}^{jk}} \approx 1 \text{ m} \]

\[ \sigma_{L_{1j}^{jk}} \approx 1 \text{ cm} \]

Too much error (5 wavelengths)!

Note that, assuming a Gaussian distribution of errors, \( \sigma_{\hat{N}_{1j}^{jk}} \approx 5 \)

guarantee only the 68% of success

Similar results with \( L_{2}, P_{2} \) measurements

As the ambiguity is constant (between cycle-slips), we would try to reduce uncertainty by averaging the estimate on time, but we will need 100 epochs to reduce noise up to \( \frac{1}{2} \) (but measurement errors are highly correlated on time!)
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**Resolving ambiguities one at a Time**

**Dual frequency** measurements: wide-laning with the Melbourne-Wübbena combination

\[ P_{1}^{jk} = \rho^{jk} + \nu_{P_{1}}^{jk} \]
\[ P_{2}^{jk} = \rho^{jk} + \nu_{P_{2}}^{jk} \]
\[ L_{1}^{jk} = \rho^{jk} + \lambda_{1} N_{1}^{jk} + \nu_{L_{1}}^{jk} \]
\[ L_{2}^{jk} = \rho^{jk} + \lambda_{2} N_{2}^{jk} + \nu_{L_{2}}^{jk} \]

\[ P_{N}^{jk} = \frac{f_{1}P_{1}^{jk} + f_{2}P_{2}^{jk}}{f_{1} + f_{2}} = \rho^{jk} + \nu_{P_{N}}^{jk} \]
\[ L_{W}^{jk} = \frac{f_{1}L_{1}^{jk} - f_{2}L_{2}^{jk}}{f_{1} - f_{2}} = \rho^{jk} + \lambda_{W} N_{W}^{jk} + \nu_{L_{W}}^{jk} \]

\[ L_{W}^{jk} - P_{N}^{jk} = \lambda_{W} N_{W}^{jk} + \nu_{P_{N}}^{jk} \rightarrow \hat{N}_{W}^{jk} = \left[ \frac{L_{W}^{jk} - P_{N}^{jk}}{\lambda_{W}} \right]_{\text{roundoff}} \]

**Fixing** \(N_{1}\) (after fixing \(N_{W}\))

\[ L_{1}^{jk} - L_{2}^{jk} = \lambda_{1} N_{1}^{jk} - \lambda_{2} N_{2}^{jk} + \nu_{L_{1} - L_{2}}^{jk} \]
\[ = (\lambda_{1} - \lambda_{2}) N_{1}^{jk} + \lambda_{2} N_{W}^{jk} + \nu_{L_{1} - L_{2}}^{jk} \]

\[ \hat{N}_{1}^{jk} = \left[ \frac{L_{1}^{jk} - L_{2}^{jk} - \lambda_{2} \hat{N}_{W}^{jk}}{\lambda_{1} - \lambda_{2}} \right]_{\text{roundoff}} \]

\[ \hat{N}_{2}^{jk} = \hat{N}_{1}^{jk} - \hat{N}_{W}^{jk} \]

\[ \hat{N}_{W}^{jk} \approx \frac{1}{\lambda_{1} - \lambda_{2}} \frac{\nu_{L_{1} - L_{2}}^{jk}}{5.4 \text{ cm}} \approx \frac{1}{4} \]

\[ \sigma_{\hat{N}_{1}^{jk}} \approx \frac{1}{\lambda_{1} - \lambda_{2}} \sqrt{2} \sigma_{\nu_{L_{1} - L_{2}}^{jk}} \frac{1.4 \text{ cm}}{5.4 \text{ cm}} \approx 1/4 \]

\[ \sigma_{\hat{N}_{W}^{jk}} \approx \frac{1}{\lambda_{1} - \lambda_{2}} \frac{\nu_{L_{W}}^{jk}}{86.2 \text{ cm}} \approx 0.8 \]

\[ N_{W} = N_{1} - N_{2} \]
\[ \lambda_{W} = \frac{c}{f_{1} - f_{2}} \approx 86.2 \text{ cm} \]
\[ \sigma_{\nu_{L_{W}}^{jk}} \approx \frac{\sigma_{\nu_{L_{1}}^{jk}}}{\sqrt{2}} \approx 71 \text{ cm} \]
\[ \sigma_{\nu_{L_{W}}^{jk}} \approx 6 \sigma_{\nu_{L_{1}}^{jk}} \approx 6 \text{ cm} \]

Now, with uncorrelated measurements from 10 epochs will reduce noise up to about \(1/4\).
Wide-Lane ambiguity fixing: IND1-IND2: 7.188m baseline: 2013 052

Master of Science in GNSS

@ J. Sanz & J.M. Juan
Once the integer ambiguities are known, the carrier phase measurements become unambiguous pseudoranges, accurate at the centimetre level (in DD), or better.

Thence, the estimation of the relative position vector is straightforward following the same approach as with pseudoranges.
Exercises:

1) Consider the wide-lane combination of carrier phase measurements

\[ L_w = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2} \]

where \( L_w \) is given in length units (i.e. \( L_i = \lambda_i \phi_i \)).

Show that the corresponding wavelength is:

\[ \lambda_w = \frac{c}{f_1 - f_2} \]

**Hint:**

\[ L_w = \lambda_w \phi_w ; \quad \phi_w = \phi_1 - \phi_2 \]

2) Assuming \( L_1, L_2 \) uncorrelated measurements with equal noise \( \sigma_L \), show that:

\[ \sigma_{L_w} = \frac{\sqrt{\gamma_{12}} + 1}{\sqrt{\gamma_{12}} - 1} \sigma_L \quad ; \quad \gamma_{12} = \left( \frac{f_1}{f_2} \right)^2 \]
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Three Frequency measurements:

We still consider the above problem of relative positioning in DD for short baselines (e.g. < 10 km) \(\rightarrow\) Ionosphere, troposphere and wind-up differential errors cancel.

<table>
<thead>
<tr>
<th>GPS</th>
<th>Frequency</th>
<th>Wavelengths</th>
<th>Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>154 x 10.23 MHz</td>
<td>(\lambda_1 = 0.190 \text{ m})</td>
<td>(\lambda_2 - \lambda_1 = 0.054 \text{ m})</td>
</tr>
<tr>
<td>L2</td>
<td>120 x 10.23 MHz</td>
<td>(\lambda_2 = 0.244 \text{ m})</td>
<td>(\lambda_W = 0.862 \text{ m})</td>
</tr>
<tr>
<td>L5</td>
<td>115 x 10.23 MHz</td>
<td>(\lambda_5 = 0.255 \text{ m})</td>
<td>(\lambda_{EW} = 5.861 \text{ m})</td>
</tr>
</tbody>
</table>

With three frequency systems, having two close frequencies it is possible to generate an extra-wide-lane signal to enable the single epoch ambiguity fixing.

We drop here the superscript \((jk)\) for simplicity

\[
L_i = \rho + \lambda_i N_i + \nu_{L_i}; \quad i = 1, 2, 5
\]

\[
L_W = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2} = \rho + \lambda_W N_W + \nu_{L_W}
\]

\[
L_{EW} = \frac{f_2 L_2 - f_5 L_5}{f_2 - f_5} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}}
\]

\[
P_N = \frac{f_1 P_1 + f_2 P_2}{f_1 + f_2} = \rho + \nu_{P_N}
\]

\[
P_{EN} = \frac{f_2 P_2 + f_5 P_5}{f_2 + f_5} = \rho + \nu_{P_{EN}}
\]

\[
\sigma_{L_W} = \sqrt{\frac{\gamma_{12} + 1}{\gamma_{12} - 1}} \sigma_L \quad \Box 5,7 \text{ cm}
\]

\[
\sigma_{L_{EW}} = \sqrt{\frac{\gamma_{25} + 1}{\gamma_{25} - 1}} \sigma_L \quad \Box 33,3 \text{ cm}
\]

\[
\sigma_{P_N} = \sqrt{\frac{\gamma_{12} + 1}{\gamma_{12} + 1}} \sigma_P \quad \Box 0,712 \text{ m}
\]

\[
\sigma_{P_{EN}} = \sqrt{\frac{\gamma_{25} + 1}{\gamma_{25} + 1}} \sigma_P \quad \Box 0,707 \text{ m}
\]

Exercise:
Justify the previous expressions for \(\sigma\).
We still consider the above problem of relative positioning in DD for short baselines (e.g. < 10 km) → Ionosphere, troposphere and wind-up differential errors cancel.

### GPS Frequency Wavelengths Combinations

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<td>( \lambda_5 = 0.255 \text{ m} )</td>
<td>( L_5 = \rho + \lambda_5 N_5 + v_{L_5} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \lambda_W = 0.862 \text{ m} )</td>
<td>( L_W = \rho + \lambda_W N_W + v_{L_W} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \lambda_{EW} = 5.861 \text{ m} )</td>
<td>( L_{EW} = \rho + \lambda_{EW} N_{EW} + v_{L_{EW}} )</td>
</tr>
</tbody>
</table>

\[ N_W = N_1 - N_2 \quad; \quad N_{EW} = N_2 - N_5 \]

\[ \hat{N}_{EW} = \left[ \frac{L_{EW} - P_{EN}}{\lambda_{EW}} \right]_{\text{roundoff}} \]

\[ \hat{N}_W = \left[ \frac{\lambda_{EW} \hat{N}_{EW} - (L_{EW} - L_W)}{\lambda_W} \right]_{\text{roundoff}} \]

\[ \hat{N}_1 = \left[ \frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2} \right]_{\text{roundoff}} \]

\[ \sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{P_{EN}} \quad 0.71 \text{ m} \quad 0.12 \]

\[ \sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W} \sigma_{L_{EW}} \quad 33.3 \text{ cm} \quad 0.39 \]

\[ \sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2\sigma_{L_1}} \quad 1.4 \text{ cm} \quad 1/4 \]

\[ \gamma_{12} = \left( \frac{f_1}{f_2} \right)^2 = \left( \frac{77}{60} \right)^2 \]

\[ \gamma_{25} = \left( \frac{f_2}{f_3} \right)^2 = \left( \frac{24}{23} \right)^2 \]

\[ \sigma_{P_N} = \sqrt{\gamma_{12} + 1} \sigma_{P_I} \quad 0.71 \text{ m} \]

\[ \sigma_{P_{EN}} = \sqrt{\gamma_{25} + 1} \sigma_{P_I} \quad 0.71 \text{ m} \]

\[ \sigma_{L_W} = \sqrt{\gamma_{12} - 1} \sigma_{L_1} \quad 5.7 \text{ cm} \]

\[ \sigma_{L_{EW}} = \sqrt{\gamma_{25} - 1} \sigma_{L_1} \quad 33.3 \text{ cm} \]
Exercise:
Repeat the previous study for the Galileo signals E1, E5b and E5a

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<tr>
<td>E5b</td>
<td>118 x 10.23 MHz</td>
<td>$\lambda_2 = 0.248$ m</td>
<td>$\lambda_W = 0.814$ m</td>
</tr>
<tr>
<td>E5a</td>
<td>115 x 10.23 MHz</td>
<td>$\lambda_3 = 0.255$ m</td>
<td>$\lambda_{EW} = 9.768$ m</td>
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</table>

\[
\hat{N}_{EW} = \left[ \frac{L_{EW} - P_{EN}}{\lambda_{EW}} \right]_{\text{roundoff}}
\]
\[
\hat{N}_W = \left[ \frac{\lambda_{EW} \hat{N}_{EW} - (L_{EW} - L_{W})}{\lambda_{W}} \right]_{\text{roundoff}}
\]
\[
\hat{N}_1 = \left[ \frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2} \right]_{\text{roundoff}}
\]

\[
\sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{P_{EN}} [ ]
\]
\[
\sigma_{\hat{N}_W} \approx \frac{1}{\lambda_{W}} \sigma_{L_{EW}} [ ]
\]
\[
\sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2 \sigma_{L_1}} [ ]
\]

\[
L_1 = \rho + \lambda_1 N_1 + \nu_{L_1}
\]
\[
L_2 = \rho + \lambda_2 N_2 + \nu_{L_2}
\]
\[
\sigma_{L_1} \approx \sigma_{L_2} \approx 1 \text{ cm}
\]
\[
\sigma_{P_1} \approx \sigma_{P_2} \approx 1 \text{ m}
\]

\[
L_W = \rho + \lambda_W N_W + \nu_{L_W}
\]
\[
P_N = \rho + \nu_{P_N}
\]
\[
L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}}
\]
\[
P_{EN} = \rho + \nu_{P_{EN}}
\]

\[
\gamma_{12} = \left( \frac{f_1}{f_2} \right)^2 = \left( \frac{77}{59} \right)^2
\]
\[
\gamma_{23} = \left( \frac{f_2}{f_3} \right)^2 = \left( \frac{118}{115} \right)^2
\]

\[
\sigma_{P_N} = \sqrt{\frac{\gamma_{12} + 1}{\gamma_{12} + 1}} \sigma_{P_1} [ ]
\]
\[
\sigma_{P_{EN}} = \sqrt{\frac{\gamma_{25} + 1}{\gamma_{25} + 1}} \sigma_{P_1} [ ]
\]

\[
\sigma_{L_W} = \frac{1}{\sqrt{\gamma_{12} - 1}} \sigma_{L_1} [ ]
\]
\[
\sigma_{L_{EW}} = \frac{1}{\sqrt{\gamma_{25} - 1}} \sigma_{L_1} [ ]
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Exercise:

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\[
\hat{N}_{EW} = \frac{L_{EW} - P_{EN}}{\lambda_{EW}} \quad \text{roundoff}
\]

\[
\hat{N}_W = \frac{\lambda_{EW} \hat{N}_{EW} - (L_{EW} - L_W)}{\lambda_W} \quad \text{roundoff}
\]

\[
\hat{N}_1 = \frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2} \quad \text{roundoff}
\]

\[
\sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{P_{EN}} \approx \frac{0.71}{9.768} \approx 0.07
\]

\[
\sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W} \sigma_{L_{EW}} \approx \frac{54.9}{81.4} \approx 0.67
\]

\[
\sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2} \sigma_{L_1} \approx \frac{1.4}{5.8} \approx 1/4
\]

\[
L_1 = \rho + \lambda_1 N_1 + \nu_{L_1}
\]

\[
L_2 = \rho + \lambda_2 N_2 + \nu_{L_2}
\]

\[
\begin{align*}
\sigma_{L_1} & \approx \sigma_{L_2} \approx 1 \text{ cm} \\
\sigma_{P_1} & \approx \sigma_{P_2} \approx 1 \text{ m}
\end{align*}
\]

\[
\begin{align*}
L_W = \rho + \lambda_W N_W + \nu_{L_W} \\
L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{L_{EW}} \\
L_{EN} = \rho + \nu_{L_{EN}}
\end{align*}
\]

\[
\begin{align*}
\gamma_{12} &= \left( \frac{f_1}{f_2} \right)^2 = \left( \frac{77}{59} \right)^2 \\
\gamma_{23} &= \left( \frac{f_2}{f_3} \right)^2 = \left( \frac{118}{115} \right)^2
\end{align*}
\]

\[
\begin{align*}
\sigma_{P_N} &= \sqrt{\gamma_{12} + 1} \sigma_{P_1} \approx 0.71 \text{ m} \\
\sigma_{P_{EN}} &= \sqrt{\gamma_{25} + 1} \sigma_{P_1} \approx 0.71 \text{ m}
\end{align*}
\]

\[
\begin{align*}
\sigma_{L_W} &= \sqrt{\gamma_{12} + 1} \sigma_{L_1} \approx 5.4 \text{ cm} \\
\sigma_{L_{EW}} &= \sqrt{\gamma_{25} + 1} \sigma_{L_4} \approx 54.9 \text{ cm}
\end{align*}
\]
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Resolving Ambiguities as a set

As a driven problem to study the ambiguity fixing, we will consider problem of differential positioning in DD for short baselines (e.g. < 10 km). To simplify, we will consider only carrier measurements at a single or dual frequency.

\[
\begin{align*}
L_{q}^{12}(t_i) &= \rho_{q}^{12}(t_i) + N_{q}^{12} + \nu_{L_{q}}^{12}(t_i) \\
L_{q}^{13}(t_i) &= \rho_{q}^{13}(t_i) + N_{q}^{13} + \nu_{L_{q}}^{13}(t_i) \\
&\vdots \\
L_{q}^{K-1}(t_i) &= \rho_{q}^{K-1}(t_i) + N_{q}^{K-1} + \nu_{L_{q}}^{K-1}(t_i)
\end{align*}
\]

\(q = 1, 2, \ldots\)

In principle, the estimation of ambiguities in this system is not a big problem if we can wait enough time and the unmodelled errors are not so large.

**Static position**

<table>
<thead>
<tr>
<th>Equations</th>
<th>Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single frequency</td>
<td>((K-1)\times n_t + 3+(K-1))</td>
</tr>
<tr>
<td>Dual frequency</td>
<td>(2(K-1)\times n_t + 3+2(K-1))</td>
</tr>
</tbody>
</table>

**Kin. position**

<table>
<thead>
<tr>
<th>Equations</th>
<th>Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single frequency</td>
<td>((K-1)\times n_t + 3\times n_t+(K-1))</td>
</tr>
<tr>
<td>Dual frequency</td>
<td>(2(K-1)\times n_t + 3\times n_t+2(K-1))</td>
</tr>
</tbody>
</table>

\(K \geq 4, \ n_t \geq 2\)

\(K \geq 5, \ n_t \geq 4\)

Each epoch brings a set of \((K-1)\) equations for short baselines (e.g. < 10 km).

\[
\begin{align*}
\rho_{jk}(t_i) &= \rho_{\hat{0}}^{jk}(t_i) - \hat{\rho}_{\hat{0}}^{jk}(t_i) \cdot \Delta r(t_i)
\end{align*}
\]

We can estimate all parameters (position and ambiguities) as a set by considering the over-dimensioned system of linear equations and solving it by the LS criterion.

\[
y(t_i) = G(t_i) \Delta r(t_i) + \lambda N + \nu
\]
Resolving Ambiguities as a set

\[
y(t_i) = G(t_i) \Delta r(t_i) + \lambda N + v(t_i)
\]

\(\tilde{K}\) \(\tilde{K} \times 3\) vector \(3\) vector \(\tilde{K}\) vector

For static positioning, considering two epochs (for instance):

\[
\begin{bmatrix}
y(t_i) \\
y(t_{i+1})
\end{bmatrix} =
\begin{bmatrix}
G(t_i) \\
G(t_{i+1})
\end{bmatrix} \Delta r + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} N +
\begin{bmatrix} v(t_i) \\ v(t_{i+1}) \end{bmatrix}
\]

In general, mixing several epochs, we will write:

\[
y = G \Delta r + \lambda A N + v
\]

Using the least-squares criterion, we can look for a real valued 3-vector \(\Delta r\) and a \(\tilde{K}\)-vector of integers \(N\) that minimizes the cost function (sum of squared residuals):

\[
c(\Delta r, N) = \|y - G \Delta r + \lambda A N\|
\]

Weighted norm can be taken as well

The problem can be easily reformulated for the kinematic case. Kalman filtering can be applied as well.
Resolving Ambiguities as a set

Different strategies can be applied:

- **To Float the ambiguities** (i.e. treating the ambiguities as real numbers).
- **To Search ambiguities** over a limited set of integers to ‘find the best solution’.
- **To solve as an Integer Least-Squares problem**.

For an observation span relatively long, e.g. one hour, the floated ambiguities would typically be very close to integers, and the change in the position solution from the float to the fixed solution should not be large.

As the observation span becomes smaller, ambiguity resolution play a more important role. But very short observation spans implies the risk of wrong ambiguity fixing, which can degrade the position solution significantly.

The performance, is thence measured by:

1. Initialization time
2. Reliability (or, correctness) of the integer estimates
Search techniques

**Strategy:**

- Define a volume to be searched
- Set up a grid within this volume
- Define a cost function (e.g. the sum of squared residuals)
- Evaluate the cost function at each grid point

Solution corresponds to the grid point with the lowest value of the cost function

<table>
<thead>
<tr>
<th>Position domain</th>
<th>Ambiguity domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ambiguity Function Method (AFM)</td>
<td>LSAS (Hatch, 1990)</td>
</tr>
<tr>
<td>ARCE</td>
<td>LAMBDA (Teunissen, 1993)</td>
</tr>
<tr>
<td></td>
<td>MLAMBDA (Chang et al. 2005)</td>
</tr>
<tr>
<td></td>
<td>OMEGA (Kim and Langley, 2000)</td>
</tr>
<tr>
<td></td>
<td>FASF (Chen and Lachapelle, 1995)</td>
</tr>
<tr>
<td></td>
<td>IP (Xu et al., 1995)</td>
</tr>
</tbody>
</table>
A conceptually simpler approach would consist on:

- Estimate the floated solution $\hat{N}$ and its uncertainty (e.g. $\hat{N}=2502347.74$ cycles, $\sigma_{\hat{N}}=0.6$ cycles)
- Define as a volume to be searched (e.g. $\pm 3 \sigma_{\hat{N}} \pm 2$ cycles) and evaluate the cost function (the RMS residuals) over the 6 ambiguities:

$$\text{RMS residuals}: 2502345, \ldots, 2502350$$

The previous search must be done for each satellite in view.

- If there are 5 satellites tracked $\Rightarrow$ 4 DD ambiguities $\Rightarrow 6^4 = 1296$ combinations
- If there are 8 satellites tracked $\Rightarrow$ 7 DD ambiguities $\Rightarrow 6^7 = 279376$ combinations

The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate. However if two or more candidates give roughly similar values of RMS, the test can not be resolute.

$\Rightarrow$ A ratio test (of 2 or 3, depending of the algorithm) between the two smallest RMS is often used to validate the test.

If the ratio is under these values, no integer solution can be determined and is better to use the floated solution.
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LAMBDA Method

Consider again the previous problem of estimating $\Delta r$, a $3$-vector of real numbers, and $N$ a $(K-1)$-vector of integers, which are solution of

$$y = G \Delta r + \lambda A N + v$$

$$\min \|y - G \Delta r - \lambda A N\|_{w_y}$$

To better exploit the internal correlations [*], we consider now the covariance $W_y = P_y^{-1}$

Let be the float solution and covariance matrix:

$$\begin{bmatrix} \Delta \hat{r} \\ \hat{N} \end{bmatrix}; \quad \text{Cov} \begin{bmatrix} \Delta \hat{r} \\ \hat{N} \end{bmatrix} = \begin{bmatrix} P_{\Delta \hat{r}} & P_{\Delta \hat{r}, \hat{N}} \\ P_{\Delta \hat{r}, \hat{N}} & P_{\hat{N}} \end{bmatrix}$$

It can be shown the following orthogonal decomposition:

$$\|y - G \Delta r - \lambda A N\|_{w_y}^2 = \|y - G \Delta \hat{r} - \lambda A \hat{N}\|_{w_y}^2 + \|\Delta r - \Delta \hat{r}(N)\|_{w_{\Delta \hat{r}(N)}}^2 + \lambda^2 \|N - \hat{N}\|_{w_{\hat{N}}}^2$$

Residual of real-valued floated solution $(\Delta \hat{r}, \hat{N})$

[*] Remember that DD measurements are correlated, as already seen.
LAMBDA Method

Thence, we have to find $\Delta r$ a 3-vector of real numbers, and $N$ a $(K-1)$-vector of integers minimizing:

$$
\|y - G \Delta r - \lambda A N\|^2_w = \|y - G \hat{\Delta r} - \lambda A \hat{N}\|^2_w + \|\Delta r - \Delta \hat{r}(N)\|^2_{\Delta \hat{r}(N)} + \lambda^2 \|N - \hat{N}\|^2_{\hat{N}}
$$

This term is irrelevant for minimization since it does not depend on $\Delta r$ and $N$

This term can be made zero for any $N$

This term must be minimized over the integers

Float solution and covariance matrix:

$$
\begin{bmatrix}
\Delta \hat{r} \\
\hat{N}
\end{bmatrix}; \quad \text{Cov} \begin{bmatrix}
\Delta \hat{r} \\
\hat{N}
\end{bmatrix} = \begin{bmatrix}
P_{\Delta \hat{r}} & P_{\Delta \hat{r}, \hat{N}} \\
P_{\Delta \hat{r}, \hat{N}} & P_{\hat{N}}
\end{bmatrix}
$$

$W_{\hat{N}}^{-1} = P_{\hat{N}} \quad W_{\Delta \hat{r}, \hat{N}} = P_{\Delta \hat{r}, \hat{N}}^{-1}

\min \|N - \hat{N}\|^2_{\hat{N}} \rightarrow \hat{N}

$$
\Delta \bar{r} = \Delta \hat{r}(\hat{N}) = \Delta \hat{r} - W_{\Delta \hat{r}, \hat{N}} W_{\hat{N}}^{-1} (\hat{N} - \hat{N})
$$

The vectors $\Delta \bar{r}$ and $\hat{N}$ are often referred to as the fixed user solution and fixed ambiguity.
The integer search: Finding the integer vector $\mathbf{N}$ that minimizes the cost function

$$c(\mathbf{N}) = \left\| \mathbf{N} - \hat{\mathbf{N}} \right\|^2_{W_N} = (\mathbf{N} - \hat{\mathbf{N}})^T W_N (\mathbf{N} - \hat{\mathbf{N}})$$

$$W_N = P_N^{-1}$$

- A diagonal $W_N$ matrix would mean that the integer ambiguity estimates are uncorrelated.
- If the weight $W_N$ matrix is diagonal, the minimizing of the cost function is trivial. The best estimate is the float ambiguity rounded to the nearest integer.

$$W_N = \begin{bmatrix}
1 / \sigma_{\hat{N}_1}^2 & 0 \\
0 & 1 / \sigma_{\hat{N}_2}^2
\end{bmatrix}$$

$$c(\mathbf{N}) = \frac{(N_1 - \hat{N}_1)^2}{\sigma_{\hat{N}_1}} + \frac{(N_2 - \hat{N}_2)^2}{\sigma_{\hat{N}_2}}$$

In practice, the estimated (float) ambiguities are highly correlated and the ellipsoidal region stretches over a wide range of cycles. This is specially the case when the measurements are limited to a single epoch or only a few epochs.

Thence, points that appears much further away from the floated solution may have lower values of cost function than those which appear nearby. In this context, the search for integer vectors can by extremely inefficient.
To improve the computational efficiency of the search, the float ambiguities can be transformed so that the elongated ellipsoid turns into a sphere-like. Thus, the search can be limited to the neighbours of the floated ambiguity.

The idea is to apply a transformation that decorrelates the ambiguities so that the matrix \( W \) becomes diagonal. \( W \) is a positive definite matrix and thence, can be always diagonalized (as a real-valued matrix) with orthogonal eigenvectors. But the problem here is that the integer ambiguities \( N \) must be transformed preserving its integer nature!

Thence, we are looking for an “integer-valued” transformation matrix \( Z \) that makes the matrix \( W \) as close as possible to a diagonal matrix (decorrelating as much as possible the ambiguities) and with similar axes (spherical).

\[
\begin{align*}
N' &= ZN \\
\hat{N}' &= Z\hat{N} \\
P_{\hat{N}'} &= ZP_{\hat{N}}Z^T
\end{align*}
\]

Moreover, the inverse of transformation matrix \( Z^{-1} \) must be also integer, to transform back the results after finding the ambiguities.

Note that \( Z, Z^{-1} \in \mathbb{Z} \implies \det(Z) = 1 \) (i.e. it is a volume-preserving transformation)
Exercise:

Show that:

$$Z, \ Z^{-1} \in \mathbb{R} \implies |\det(Z)| = 1$$

That is, $Z$ is a volume-preserving transformation.
Decorrelation: Computing the Z-transform

The following conditions must be fulfilled:

1. $Z$ must have integer entries
2. $Z$ must be invertible and have integer entries
3. The transformation $Z$ must reduce the product of all ambiguity variances.

Note that $Z, Z^{-1} \in \mathbb{Z} \Rightarrow |\det(Z)| = 1$
(i.e. it is a volume-preserving transformation)

Gauss manipulation over matrix $P = W^{-1}$ can be applied to find-out the matrix $Z$.

$P_N = \begin{bmatrix} p_{\hat{N}_1\hat{N}_1} & p_{\hat{N}_1\hat{N}_2} \\ p_{\hat{N}_1\hat{N}_2} & p_{\hat{N}_2\hat{N}_2} \end{bmatrix}$

$Z_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$

$\Rightarrow$ Transforms $N_2$ ($N_1$ remains unchanged)

$\alpha_i = -\text{int} \left[ \frac{p_{\hat{N}_1\hat{N}_2}}{p_{\hat{N}_1\hat{N}_1}} \right]$

$Z_2 = \begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ Transforms $N_1$ ($N_2$ remains unchanged)

Note: Inverse matrices have also integer entries

$Z_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha_1 & 1 \end{bmatrix}$

$Z_2^{-1} = \begin{bmatrix} 1 & -\alpha_i \\ 0 & 1 \end{bmatrix}$

Start transforming first the element with largest variance.
Gauss manipulation over matrix $P = W^{-1}$ can be applied to find-out the matrix $Z$

$$P_\hat{N} = \begin{bmatrix} p_{\hat{N}_1\hat{N}_1} & p_{\hat{N}_1\hat{N}_2} \\ p_{\hat{N}_1\hat{N}_2} & p_{\hat{N}_2\hat{N}_2} \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix} \Rightarrow \text{Transforms } N_2 \ (N_1 \text{ remains unchanged})$$

$$Z_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{Transforms } N_1 \ (N_2 \text{ remains unchanged})$$

Example:

$$\hat{N} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} \quad P_\hat{N} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}$$

Step 1:

$$Z_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \alpha_2 = -\text{int}[38.4 / 28.0] = -1$$

$$P_{\hat{N}'} = Z_2 P_\hat{N} Z_2^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix}$$

We transform first the element with largest variance (in this case $N_1$)

The half, at most!

Step 2:

$$Z_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \alpha_1 = -\text{int}[10.4 / 4.6] = -2$$

$$P_{\hat{N}''} = Z_1 P_{\hat{N}'} Z_1^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

In general, to increase the number of small off-diagonal elements, we have to transform first the elements with largest variance.
\[ Z_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \]

\[ Z_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \]

\[ \mathbf{P}_N = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \]

\[ \mathbf{P}_{N'} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix} \]

\[ \mathbf{P}_{N''} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix} \]

\[ Z = Z_1 Z_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \]

\[ \mathbf{P}_N'' = Z \mathbf{P}_N Z^T = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix} \]
Example:

\[
\hat{N} = \begin{bmatrix}
1.05 \\
1.30
\end{bmatrix} \quad \mathbf{P}_{\hat{N}} = \begin{bmatrix}
53.4 & 38.4 \\
38.4 & 28.0
\end{bmatrix} \quad \mathbf{P}_{\hat{N}''} = \begin{bmatrix}
4.6 & 1.2 \\
1.2 & 4.8
\end{bmatrix}
\]

\[
\mathbf{Z} = \begin{bmatrix}
1 & -1 \\
-2 & 3
\end{bmatrix}
\]

\[
\hat{N}'' = \mathbf{Z} \hat{N} = \begin{bmatrix}
1 & -1 \\
-2 & 3
\end{bmatrix} \begin{bmatrix}
1.05 \\
1.30
\end{bmatrix} = \begin{bmatrix}
-0.25 \\
1.80
\end{bmatrix}
\]

\[
\hat{N}'' = \text{int} \begin{bmatrix}
-0.25 \\
1.80
\end{bmatrix} = \begin{bmatrix}
0 \\
2
\end{bmatrix}
\]

\[
\hat{N} = \mathbf{Z}^{-1} \begin{bmatrix}
0 \\
2
\end{bmatrix} = \begin{bmatrix}
3 & 1 \\
2 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
2
\end{bmatrix} = \begin{bmatrix}
2 \\
2
\end{bmatrix}
\]
Let $\mathbf{P}$ be a symmetric and positive-definite matrix:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\mathbf{P}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2} \left( p_{11} + p_{22} + w \right)$$

$$\lambda_2 = \frac{1}{2} \left( p_{11} + p_{22} - w \right)$$

$$w = \sqrt{(p_{11} - p_{22})^2 + 4 p_{12}^2}$$

$$\tan 2\phi = \frac{2 p_{12}}{p_{11} - p_{22}}$$

Example:

$$\mathbf{P}_{\hat{N}} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix}$$

$$\sqrt{\lambda_1} = \sqrt{81.14} = 9.0$$

$$\sqrt{\lambda_2} = \sqrt{0.25} = 0.5$$

$$\tan 2\phi = 3.02 \Rightarrow \phi = 35.85$$
Consider again the previous problem of estimating $\Delta \mathbf{r}$, a 3-vector of real numbers, and $\mathbf{N}$ a $(K-1)$-vector of integers, which are solution of

$$y = \mathbf{G} \Delta \mathbf{r} + \lambda \mathbf{A} \mathbf{N} + \mathbf{v}$$

The solution comprises the following steps:

1. Obtain the float solution and its covariance matrix:

$$\begin{bmatrix} \Delta \hat{\mathbf{r}} \\ \hat{\mathbf{N}} \end{bmatrix} ; \begin{bmatrix} \mathbf{P}_{\Delta \hat{\mathbf{r}}} & \mathbf{P}_{\Delta \hat{\mathbf{r}}, \hat{\mathbf{N}}} \\ \mathbf{P}_{\Delta \hat{\mathbf{r}}, \hat{\mathbf{N}}} & \mathbf{P}_{\hat{\mathbf{N}}} \end{bmatrix}$$

2. Find the integer vector $\mathbf{N}$ which minimizes the cost function

$$c(\mathbf{N}) = \| \mathbf{N} - \hat{\mathbf{N}} \|^2_{\mathbf{W}_{\hat{\mathbf{N}}}} = (\mathbf{N} - \hat{\mathbf{N}})^T \mathbf{W}_{\hat{\mathbf{N}}} (\mathbf{N} - \hat{\mathbf{N}})$$

$$\mathbf{W}_{\hat{\mathbf{N}}} = \mathbf{P}_{\hat{\mathbf{N}}}^{-1}$$

   a) **Decorrelation:** Using the $\mathbf{Z}$ transform, the ambiguity search space is re-parametrized to decorrelate the float ambiguities.

   b) **Integer ambiguities estimation** (e.g. using sequential conditional least-squares adjustment, together with a discrete search strategy).

   c) Using the $\mathbf{Z}^{-1}$ transform, the ambiguities are transformed to the original ambiguity space.

3. Obtain the ‘fixed’ solution $\Delta \mathbf{r}$, from the fixed ambiguities $\mathbf{N}$.

$$\mathbf{y} - \lambda \mathbf{A} \mathbf{N} = \mathbf{G} \Delta \mathbf{r} + \mathbf{v}$$
b) **Integer ambiguities estimation**

Several approach can be applied:

- Integer rounding
- Integer bootstrapping
- Integer Least-Squares
- ….

**Comment:**

*In principle, the previous transformation \( Z \) is not required by the estimation concept; it is only to achieve considerable gain in speed in the computation process [RD-5].*

b1) **Integer rounding**

This is the simplest way.

Just to round-up the ambiguity vector entries to its nearest integer

\[
\tilde{N} = (\text{int}(N_1), \ldots, \text{int}(N_K))
\]

For instance, in the previous example:

\[
\tilde{N}'' = \text{int} \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]
b2) Integer bootstrapping (from [RD-6])

It makes use of integer rounding, but it takes some of the correlations between the ambiguities into account.

1. We start with the most precise ambiguity (here we will assume $N_n$).
2. Then, the remaining float ambiguities are corrected taking into account their correlation with the last ambiguity.

\[
\tilde{N}_n = \text{int} \left[ \hat{N}_n \right] \\
\tilde{N}_{n-1} = \text{int} \left[ \hat{N}_{n-1|n} \right] = \text{int} \left[ \hat{N}_{n-1} - \sigma_{\hat{N}_{n-1},\hat{N}_n} \sigma^{-2}_{\hat{N}_n} (\hat{N}_n - \tilde{N}_n) \right] \\
\vdots \\
\tilde{N}_1 = \text{int} \left[ \hat{N}_{n|I} \right] = \text{int} \left[ \hat{N}_1 - \sum_{i=2}^{n} \sigma_{\hat{N}_1,\hat{N}_i} \sigma^{-2}_{\hat{N}_i} (\hat{N}_i - \tilde{N}_i) \right]
\]

\[\hat{N}_{i|I}\] Stands for the $i$-th ambiguity obtained through a conditioning of the previous $I = \{i+1, \ldots, n\}$ sequentially rounded ambiguities.

Using the triangular decomposition

\[P_{\hat{N}} = L^T D L \]

\[l_{ij} = \sigma_{\hat{N}_j,\hat{N}_{i|I}} \sigma^{-2}_{\hat{N}_{i|I}} \]
\[ N_1 = \hat{N}_1 - \sigma_{\hat{N}_2, \hat{N}_1} \sigma_{\hat{N}_2}^{-2} \left( \hat{N}_2 - N_2 \right) \]

Figure 2.2: Principle and 2D pull-in regions for integer bootstrapping: parallelograms.

\[ \tilde{N}_2 = \text{nint} \left[ \hat{N}_2 \right] = 0 \]
\[ \tilde{N}_1 = \text{nint} \left[ \hat{N}_{1|2} \right] = \text{nint} \left[ \hat{N}_1 - \sigma_{\hat{N}_2, \hat{N}_1} \sigma_{\hat{N}_2}^{-2} \left( \hat{N}_2 - \tilde{N}_2 \right) \right] = 1 \]
b3) Integer Least Squares (ISL) (from [RD-6])

1. The target to find the integer vector $\mathbf{N}$ which minimizes the cost function

$$c(\mathbf{N}) = \| \mathbf{N} - \mathbf{\hat{N}} \|^2_{\mathbf{P}^{-1}} = (\mathbf{N} - \mathbf{\hat{N}})^T \mathbf{P}^{-1}_N (\mathbf{N} - \mathbf{\hat{N}})$$

$$\mathbf{W}_N = \mathbf{P}^{-1}_N$$

2. The integer minimiser is obtained through a search over the integer grid points on the $n$-dimensional hyper-ellipsoid:

$$\left( \mathbf{N} - \mathbf{\hat{N}} \right)^T \mathbf{P}^{-1}_N (\mathbf{N} - \mathbf{\hat{N}}) \leq \chi^2$$

- Where $\chi^2$ determines the size of search region.
- The solution is the integer grid point $\mathbf{N}$, inside the ellipsoid, giving the minimum value of cost function $c(\mathbf{N})$.

Using the triangular decomposition:

$$\mathbf{P}_N = \mathbf{L}^T \mathbf{D} \mathbf{L}$$

where:

$$d_i = \sigma^{-2}_{\hat{N}_i}$$

$$l_{ij} = \sigma^{-2}_{\hat{N}_j,\hat{N}_i}$$

$$p_{ij} = \sigma^{-2}_{\hat{N}_j,\hat{N}_i}$$

Defining:

$$\tilde{\mathbf{N}} = \mathbf{N} - \mathbf{L}^T (\mathbf{N} - \mathbf{\hat{N}}) \rightarrow \mathbf{L}^T (\tilde{\mathbf{N}} - \mathbf{N}) = (\mathbf{\hat{N}} - \mathbf{N})$$

$$c(\mathbf{N}) = \frac{(N_1 - \tilde{N}_1)^2}{d_1} + \frac{(N_2 - \tilde{N}_2)^2}{d_2} + \ldots + \frac{(N_n - \tilde{N}_n)^2}{d_n} \leq \chi^2$$
\((\mathbf{N} - \tilde{\mathbf{N}})^T \mathbf{D}^{-1} (\mathbf{N} - \tilde{\mathbf{N}}) \leq \chi^2\)

But \(\tilde{N}_i\) depends on \(\tilde{N}_{i+1}, \ldots, \tilde{N}_n\).

\(\mathbf{L}^T (\tilde{\mathbf{N}} - \mathbf{N}) = (\tilde{\mathbf{N}} - \mathbf{N})\)

\[
c(\mathbf{N}) = \frac{\left(N_1 - \tilde{N}_1\right)^2}{d_1} + \frac{\left(N_2 - \tilde{N}_2\right)^2}{d_2} + \cdots + \frac{\left(N_n - \tilde{N}_n\right)^2}{d_n} \leq \chi^2
\]

\[
\tilde{N}_n = \tilde{N}_n
\]

\[
\tilde{N}_i = \tilde{N}_i + \sum_{j=i+1}^{n} \left(N_j - \tilde{N}_i\right) l_{ji}; \quad i = n-1, n-2, \ldots, 1
\]

**Search region bounds:**

\[
\tilde{N}_n - d_n^{1/2} \chi \leq N_n \leq \tilde{N}_n + d_n^{1/2} \chi
\]

\[
\tilde{N}_{n-1} - d_{n-1}^{1/2} \left(\chi^2 - \left(N_n - \tilde{N}_n\right)^2 d_n\right)^{1/2} \leq N_{n-1} \leq \tilde{N}_{n-1} + d_{n-1}^{1/2} \left(\chi^2 - \left(N_n - \tilde{N}_n\right)^2 d_n\right)^{1/2}
\]

\[
\vdots
\]

\[
\tilde{N}_1 - d_1^{1/2} \left(\chi^2 - \sum_{j=2}^{n} \left(N_j - \tilde{N}_j\right)^2 d_j\right)^{1/2} \leq N_1 \leq \tilde{N}_1 + d_1^{1/2} \left(\chi^2 - \sum_{j=2}^{n} \left(N_j - \tilde{N}_j\right)^2 d_j\right)^{1/2}
\]

**Acceptance test:** The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate. However if two or more candidates give roughly similar values of RMS, the test can not be resolute. ➔ A ratio test (of 2 or 3, depending of the algorithm) between the two smallest RMS is often used to validate the test.
Ellipsoid size: selecting the candidates for the acceptance test

The size of the ellipsoidal search region \((N - \tilde{N})^T D^{-1} (N - \tilde{N}) \leq \chi^2\) is controlled by \(\chi^2\). Therefore, the performance of the search process is highly dependent on \(\chi^2\):

- A small \(\chi^2\) may result in an ellipsoidal region that fails to contain the solution.
- A too large value for \(\chi^2\) may result in high time-consuming for the search process.

Search with enumeration: When the number of required candidates is at most \(n+1\) (with \(n=\text{dim}(N)\)), the following procedure can be applied to set the value \(\chi^2\):

- The best determined ambiguity is rounded to its nearest integer. The remaining ambiguities are then rounded using their correlations with the first ambiguity:

\[
\tilde{N}_n = \text{nint} \left[ \hat{N}_n \right]
\]

\[
\tilde{N}_{n-1} = \text{nint} \left[ \hat{N}_{n-1|n} \right] = \text{nint} \left[ \hat{N}_{n-1} - \sigma_{\hat{N}_{n-1},\hat{N}_n} \sigma_{\hat{N}_n}^{-2} (\hat{N}_n - \tilde{N}_n) \right]
\]

\[
\vdots
\]

\[
\tilde{N}_1 = \text{nint} \left[ \hat{N}_{1|I} \right] = \text{nint} \left[ \hat{N}_1 - \sum_{i=2}^{n} \sigma_{\hat{N}_1,\hat{N}_i\hat{N}_i} \sigma_{\hat{N}_i}^{-2} (\hat{N}_i - \tilde{N}_i) \right]
\]

- In each step of the conditional rounding procedure, two candidates are taken: The nearest and second-nearest, and conditional rounding is proceeded in both cases.
- If \(p\) candidates are requested, the values of cost function \(c(N)\) are ordered in ascending order and \(\chi^2\) is chosen equal to the \(p\)-th value.

If more than \(n+1\) candidates are requested, the volume of the search ellipsoid can be used ([RD-6]).
**Search with shrinking technique: practical example**

This is an alternative to the previous strategy, based on shrinking the search ellipsoid during the process of finding the candidates.

In the next example, we have to choose 6 candidates:

First candidate is the bootstrapped solution.

Round \textquotedblleft Ambiguity-2\textquotedblright{} to the second near integer and round the new conditional estimate for \textquotedblleft Ambiguity-1\textquotedblright{}.

The other 5 candidates are found by choosing the conditional \textquotedblleft Ambiguity-1\textquotedblright{} to the 2nd, 2rd, 4th and 5th nearest integers.

The candidate with the largest $\chi^2$ is removed.

Example and pictures from [RD-6]
Thence, the best 6 candidates are found (in the ISL sense). The one with the smallest cost function $c(N)$ value is the actual ISL solution.
Acceptance Test

The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate.

However if two or more candidates give roughly similar values of RMS, the test can not be resolutive.

→ A ratio test (of 2 or 3, depending on the algorithm) between the two smallest RMS is often used to validate the test.

If the ratio is under these values, no integer solution can be determined and is better to use the floated solution.

\[
RMS = \| \mathbf{N} - \hat{\mathbf{N}} \|_{P_N^{-1}} = \sqrt{ (\mathbf{N} - \hat{\mathbf{N}})^T P_N^{-1} (\mathbf{N} - \hat{\mathbf{N}}) }
\]
Examples with MATLAB (octave)

LAMBDA software package
Matlab implementation, Version 3.0

Sandra Verhagen and Bofeng Li

Note: This document uses the transposed matrix $Z^T$, but the principle is the same.
Examples with MATLAB (octave)

load large → Q, a

\[ [Qz, Zt, Lz, Dz, az, iZ] = \text{decorrel}(Q, a); \]

\[ Q \equiv P_N = W_N^{-1} \]

\[ Qz = Lz' \cdot \text{diag}(Dz) \cdot Lz \]
\[
\begin{align*}
\mathbf{Q}_z &= \mathbf{L}_z' \mathbf{D}_z \mathbf{L}_z \\
\mathbf{a}_z &= \mathbf{Z} \mathbf{a} \\
\mathbf{a} &= \mathbf{Z}^{-1} \mathbf{a}_z \\
\mathbf{Z} &= \begin{bmatrix}
3 & 0 & -4 & -3 & -5 & -4 & -4 & 2 & -2 & 1 & -3 & 1 \\
-0 & -1 & 1 & -1 & -2 & 4 & 4 & -3 & 4 & 1 & 0 & 1 \\
3 & 5 & -2 & -2 & 1 & -1 & -2 & 1 & -1 & -4 & -1 & -1 \\
-5 & -2 & 3 & 2 & 4 & 3 & -3 & -2 & -2 & 1 & -3 & -1 \\
4 & 5 & 1 & 4 & 2 & 6 & 5 & 2 & -4 & 1 & 2 & -4 \\
-8 & -4 & 1 & 0 & 0 & -3 & 2 & 3 & 2 & -1 & -0 & 4 \\
4 & -7 & -0 & 1 & 0 & -4 & -1 & -7 & 3 & -5 & -1 & 2 \\
2 & -1 & -8 & -1 & 2 & -4 & 1 & 2 & -4 & 2 & 2 & -2 \\
-3 & 2 & 3 & 10 & -8 & -2 & -5 & 0 & -4 & 1 & -4 & 0 \\
-1 & 6 & 8 & -1 & 2 & 1 & 2 & 7 & 3 & -2 & 6 & 1 \\
-8 & 7 & -8 & 3 & -6 & -1 & 1 & 0 & 0 & 3 & -1 & -1 \\
8 & 1 & 6 & -3 & 5 & 4 & -5 & -3 & 0 & -0 & 1 & -3
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\mathbf{Q}_z &= \mathbf{Z} \mathbf{Q} \mathbf{Z}' \\
\mathbf{Q} &= \mathbf{Z}^{-1} \mathbf{Q}_z \mathbf{Z}^{-1} \\
\mathbf{L}, \mathbf{D} &= \text{ldldecom}(\mathbf{Q}) \\
\end{align*}
\]
**Integer rounding**

```
round(a)
```

```
[ -28491  65753  38830  5004 -29196 -298 -22201  51236  30258  3899 -22749 -159]
```

**Decorrelation + Integer rounding**

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=round(az);
afixed=iZ*azfixed
```

```
[ -28537  65473  38692  4939 -29228 -504 -22237  51018  30150  3849 -22774 -320]
```

**Decorrelation + bootstrapping**

```
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a)
azfixed=bootstrap(az,Lz);
afixed=iZ*azfixed
```

```
[ -28451  65749  38814  5025 -29165 -278 -22170  51233  30245  3916 -22725 -144]
```
Decorrelation + bootstrapping

\[ [Q_z, Z_t, L_z, D_z, \alpha, i_Z] = \text{decorrel}(Q, a) \]
\[
\text{azfixed} = \text{bootstrap}(az, L_z);
\]
\[
\text{afixed} = i_Z \times \text{azfixed}
\]

\[
\begin{array}{cccccccccc}
\end{array}
\]

Decorrelation + ILS with enumeration search

\[ [Q_z, Z_t, L_z, D_z, \alpha, i_Z] = \text{decorrel}(Q, a); \]

\[ [\text{azfixed}, \text{sqnorm}] = \text{lsearch}(az, L_z, D_z, 6); \]

\[ \text{afixed} = i_Z \times \text{azfixed} \]

\[
\begin{array}{cccccccccc}
-28451 & 65749 & 38814 & 5025 & -29165 & -278 & -22170 & 51233 & 30245 & 3916 & -22725 & -144 \rightarrow 15.0 \\
-28279 & 65862 & 38805 & 5170 & -29061 & -192 & -22036 & 51321 & 30238 & 4029 & -22644 & -77 \rightarrow 31.6 \\
-28727 & 65935 & 39032 & 4844 & -29337 & -178 & -22385 & 51378 & 30415 & 3775 & -22859 & -66 \rightarrow 33.9 \\
-28546 & 66062 & 39027 & 4998 & -29228 & -83 & -22244 & 51477 & 30411 & 3895 & -22774 & 8 \rightarrow 34.5 \\
-28229 & 65518 & 38583 & 5197 & -29056 & -500 & -21997 & 51053 & 30065 & 4050 & -22640 & -317 \rightarrow 34.7 \\
-28365 & 65586 & 38683 & 5084 & -29124 & -418 & -22103 & 51106 & 30143 & 3962 & -22693 & -253 \rightarrow 35.5
\end{array}
\]
Decorrelation + bootstrapping

\[
\begin{align*}
\{Q_z, Z_t, L_z, D_z, az, iZ\} &= \text{decorrel}(Q, a) \\
\text{azfixed} &= \text{bootstrap}(az, L_z) \\
\text{afixed} &= iZ \cdot \text{azfixed}
\end{align*}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]

Decorrelation + ILS with search-and-shrink

\[
\begin{align*}
\{Q_z, Z_t, L_z, D_z, az, iZ\} &= \text{decorrel}(Q, a) \\
[\text{azfixed}, \text{sqnorm}] &= \text{ssearch}(az, L_z, D_z, 6); \\
\text{afixed} &= iZ \cdot \text{azfixed}
\end{align*}
\]

\[
\begin{array}{ccccccccccc}
\text{c(N)}
\end{array}
\]

\[
\begin{bmatrix}
-28279 & 65862 & 38805 & 5170 & -29061 & -192 & -22036 & 51321 & 30238 & 4029 & -22644 & -77 \\
-28546 & 66062 & 39027 & 4998 & -29228 & -83 & -22244 & 51477 & 30411 & 3895 & -22774 & 8 \\
\end{bmatrix}
\]
References


Least-Squares Ambiguity Search technique

This technique requires an approximate solution, which can be obtained from code range measurements. The search area can be defined by surrounding the approximate position by a $3\sigma$ region (i.e. $\Delta \hat{r} \pm \delta$).

$$y = G \Delta r + \lambda N + v \quad \rightarrow \quad \hat{N} = \frac{1}{\lambda} (y - G \Delta \hat{r})$$

\(K-1\) equations with \(3+(K-1)\) unknowns.

Then, given the 3-D position, the \((K-1)\) integer ambiguities can be resolved automatically.

That is, the \((K-1)\) integer ambiguities are constrained to three degrees of freedom.
The technique is based on exploiting the constrains on the integer ambiguities:

- The tracked satellites into two groups: **4 sat.** (with good DOP) + **N-4 sat.**

- The primary group of **4 sat.** is used determine the possible ambiguity sets $N(k)$ (given an initial position estimate and its associated uncertainty $\Delta \hat{r} \pm \sigma$)

$$\Delta \hat{r} \pm \sigma \rightarrow N(k) = \frac{1}{\lambda} \left( y - G \left( \Delta \hat{r} + \delta_{(k)} \right) \right) \rightarrow N(k)$$

Then, the corresponding position estimates $\Delta \hat{r}_{(k)}$ are computed.

$$N(k) \rightarrow \Delta \hat{r}_{(k)} = \left( G^T G \right)^{-1} G \left[ y - \lambda N(k) \right]$$

- The remaining $N-4$ secondary satellites are used to eliminate candidates of the possible ambiguity sets:
  - Each position estimate is checked against measurements from the secondary group of **N-4 satellites.**
  - With the correct position estimate, the difference between the measured and computed carrier phase should be “close to an integer” for each satellite pair.

$$\frac{1}{\lambda} \left( y - G \Delta \hat{r}_{(k)} \right) - N(k) \in \square ?$$
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